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A SIMPLE PROOF OF THE BERNOULLICITY OF ERGODIC AUTOMORPHISMS ON COMPACT ABELIAN GROUPS

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ABSTRACT

Recently it was proved by D. Lind, and G. Miles and K. Thomas that every ergodic automorphism of a compact metric abelian group is Bernoullian. They reduce the problem to the finite-dimensional compact connected abelian group (solenoidal group), and then they use difficult methods in proving the case. By using ideas of Y. Katznelson we can give a proof, which is much simpler than the other extant proofs, for the solenoidal case.

§1. Ergodic automorphisms of solenoidal groups

In this section we shall prove the following

THEOREM 1. *If σ is an ergodic automorphism of a solenoidal group X, then (X,σ) is Bernoullian.*

PROOF. Let (G,γ) denote the dual of (X,σ) ($(\gamma g)(x)=g(\sigma x)$, $g\in G$ and $x\in X$). If $\operatorname{rank}(G)=r<\infty$, then G is imbedded in $\mathbf{Q}^r$ (denoting the r-dimensional vector space over the rational field $\mathbf{Q}$), and there is an extension of γ on $\mathbf{Q}^r$ which we denote by the same symbol. Let $(\bar{T}^r,\sigma)$ be the dual of $(\mathbf{Q}^r,\gamma)$, then it is easy to see that $(\bar{T}^r,\sigma)$ is ergodic, since γ has no finite orbits except the identity of $\mathbf{Q}^r$. If we hold the following Proposition 1, then $(\bar{T}^r,\sigma)$ is Bernoullian. Since (X,σ) is isomorphic to $(\bar{T}^r/\bar{T}(G),\sigma)$ where $\bar{T}(G)$ is the annihilator of G in $\bar{T}^r$, by Ornstein's theorem [17] we get the conclusion of Theorem 1.

PROPOSITION 1. *If $(\bar{T}^r,\sigma)$ is ergodic, then it is Bernoullian.*

We identify γ with the matrix of $GL(r,\mathbf{Q})$ corresponding to γ. It is known (cf. see p. 397 of [9]) that $\mathbf{Q}^r$ admits a direct sum splitting $\mathbf{Q}^r=\mathbf{Q}^{r_1}\oplus\mathbf{Q}^{r_2}\oplus\cdots\oplus\mathbf{Q}^{r_k}$

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where $\mathbf{Q}^{r_i} = \mathrm{span}_{\mathbf{Q}}\{\gamma^j f_i : j \in \mathbf{Z}\}$ for some $f_i \in \mathbf{Q}^r$ $(1 \leqq i \leqq k)$. Hence $(\bar{T}^r, \sigma)$ splits into the direct sum

$$(\bar{T}^r, \sigma) = (\bar{T}^{r_1}, \sigma) \oplus (\bar{T}^{r_2}, \sigma) \oplus \cdots \oplus (\bar{T}^{r_k}, \sigma)$$

of ergodic systems $(\bar{T}^{r_i}, \sigma)$ where each $\bar{T}^{r_i}$ is a σ-invariant subgroup of $\bar{T}^r$. Therefore it is enough to show that each $(\bar{T}^{r_i}, \sigma)$ is Bernoullian.

We shall show the following proposition which implies Proposition 1.

PROPOSITION 2. *Assume that all the eigenvalues of γ are not roots of unity and $\mathbf{Q}^r = \mathrm{span}_{\mathbf{Q}}\{\gamma^j f : j \in \mathbf{Z}\}$ for some $f \in \mathbf{Q}^r$. Then the dual $(\bar{T}^r, \sigma)$ of $(\mathbf{Q}^r, \gamma)$ is Bernoullian.*

The proof is similar to that of Y. Katznelson [6], so we shall sketch the proof here. See [6] for details.

Let $\langle f \rangle$ denote the cyclic group generated by f and $\bar{T}(\langle f \rangle)$ denote the annihilator of $\langle f \rangle$ in $\bar{T}^r$, then $\bar{T}^r / \bar{T}(\langle f \rangle)$ is isomorphic to the one-dimensional torus $[0, 1)$ (mod 1). Let n be a fixed positive integer and $\mathcal{P}_n$ be the partition of $\bar{T}^r / \bar{T}(\langle f \rangle)$ corresponding to the partition of $[0, 1)$ into the intervals with the same lengths $1/2^n$. Let $\pi : \bar{T}^r \to \bar{T}^r / \bar{T}(\langle f \rangle)$ be the canonical projection, then $\pi^{-1}(\mathcal{P}_n)$ is a partition of $\bar{T}^r$.

We denote by $\varphi(\cdot, g)$ $(g \in \mathbf{Q}^r)$ *a character of* $\bar{T}^r$, and define for $p \in \mathcal{P}_n$ and a positive integer m

$$f_{m,p}(t) = (1 + m^{-2}) \int_{\pi^{-1}p} \sum_{k=-m^{20}}^{m^{20}} 2^{-1}\{1 - |k| [m^{20} + 1]^{-1}\} \varphi(t - s, kf) d\mu(s)$$

which corresponds to the Fejér sum of order m^{20} of the characteristic function of a subset of $[0, 1)$ multiplied by $1 + m^{-2}$.

A typical atom A in a partition

$$\mathcal{A} = \bigvee_{m=0}^{K^2} \sigma^{-m} \pi^{-1}(\mathcal{P}_n)$$

has a form $A = \bigcap_{m=0}^{K^2} \sigma^{-m} \pi^{-1} p_{j_m}$ $(j_m = 1, 2, \cdots, 2^n)$. For a positive integer J, define on $\bar{T}^r$ a non-negative function $\varphi_A(t) = \Pi_{m=0}^{K^2} f_{m+J, p_{j_m}}(\sigma^m t)$. Then we get

$$\varphi_A(t) = \prod_{m=0}^{K^2} \sum_{k=-(m+J)^{20}}^{(m+J)^{20}} C_{k,m} \varphi(\sigma^m t, kf)$$

in which

$$C_{k,m} = 2^{-1}[1 + (m + J)^{-2}]\{1 - |k| [(m + J)^{20} + 1]^{-1}\} \int_{\pi^{-1}p_{j_m}} \varphi(-s, kf) d\mu(s).$$

It is easy to check that $\varphi_A(t)$ is expressed in the form

$$\varphi_A(t) = \sum_{\{k_m\}} C_{\{k_m\}} \varphi\left(t, \sum_{m=0}^{K^2} k_m \gamma^m f\right)$$

where each $C_{\{k_m\}}$ is some constant. Similarly, define $\varphi_B(t) = \Pi_{m=0}^{M} f_{m+J, p_{j_m}}(\sigma^{m+K^2+K} t)$ for an atom $B = \sigma^{-K^2-K} \bigcap_{m=0}^{M} \sigma^{-m} \pi^{-1} p_{j_m}$ in a partition

$$\mathcal{B} = \sigma^{-K^2-K} \bigvee_{m=0}^{M} \sigma^{-m} \pi^{-1}(\mathcal{P}_n).$$

Then we obtain easily

$$\varphi_B(t) = \sum_{\{k_{m+K^2+K}\}} D_{\{k_{m+K^2+K}\}} \varphi\left(t, \gamma^{K^2+K} \sum_{m=0}^{M} k_{m+K^2+K} \gamma^m f\right)$$

where each $D_{\{k_{m+K^2+K}\}}$ is some constant.

Let $\varepsilon > 0$, then we can calculate (see p. 190 of [6]) that there exist an integer $J = J(\mathcal{P}_n, \varepsilon) > 0$ and a Borel set $E \subset \bar{T}^r$ with $\mu(E) < \varepsilon^2$ such that for all $K > 0$ and all $M > 0$

$$\varphi_A(t) \geqq 1 \quad \text{on } A \setminus E, \qquad \varphi_B(t) \geqq 1 \quad \text{on } B \setminus E,$$

$$\sum_A \varphi_A(t) \leqq 1 + \varepsilon^2, \qquad \sum_B \varphi_B(t) \leqq 1 + \varepsilon^2.$$

By Proposition 3 in the next section together with the Fourier expansions of $\varphi_A(t)$ and $\varphi_B(t)$, we see that there is an integer $K_1 > 0$ such that for all $K \geqq K_1$ and all $M > 0$, the only frequency which is common to $\varphi_A(t)$ and $\varphi_B(t)$ is zero. Thus, from the orthogonality of $\{\varphi(\cdot, g) : g \in G\}$, it follows that for $K \geqq K_1$ and $M > 0$,

$$\int \varphi_A \varphi_B d\mu = \int \varphi_A d\mu \int \varphi_B d\mu.$$

Hence $\pi^{-1}(\mathcal{P}_n)$ is an almost weak Bernoulli partition for $(\bar{T}^r, \sigma)$ (see lemma 1 of [6]). Denote $K_f = \Sigma_{-\infty}^{\infty} \gamma^j \langle f \rangle$ and by $\bar{T}(K_f)$ the annihilator of K_f in $\bar{T}^r$. Since each element of the partition $\bigvee_{j=-\infty}^{\infty} \sigma^j \{ \bigvee_{n=1}^{\infty} \pi^{-1}(\mathcal{P}_n)\}$ is a coset of $\bar{T}(K_f)$, $(\bar{T}^r / \bar{T}(K_f), \sigma)$ is Bernoullian. Let $\tilde{\kappa}_n : \mathbf{Q}^r \to \mathbf{Q}^r$ be an isomorphism defined by $\tilde{\kappa}_n(g) = (1/n!)g$ for $n \geqq 1$, then $\tilde{\kappa}_n(K_f) \nearrow \mathbf{Q}^r$ as $n \to \infty$ and hence $\bar{T}(\tilde{\kappa}_n(K_f)) \searrow \{0\}$ as $n \to \infty$. Since $(\bar{T}^r / \bar{T}(\tilde{\kappa}_n K_f), \sigma)$ is Bernoullian for any $n \geqq 1$, $(\bar{T}^r, \sigma)$ is Bernoullian by Ornstein's theorem [16].

§2. Non-singular matrices in $\mathbf{Q}^r$

The proof of Proposition 3 was improved to the present one by I. Kubo and the author [2]. Let $\mathbf{Q}^r$ be the r-dimensional vector space over $\mathbf{Q}$ and γ be in $GL(r,\mathbf{Q})$. Consider γ as an action on $\mathbf{Q}^r$ and assume $\mathbf{Q}^r = \mathrm{span}_{\mathbf{Q}}\{\gamma^j f : j \in \mathbf{Z}\}$ for some vector $f \neq 0$. It is easy to see that the characteristic polynomial $h(x)$ of the matrix γ is itself the minimal polynomial over $\mathbf{Q}$. We denote by $p(x)$ the primitive polynomial such that $p(x) = ch(x)$ with a natural number c.

LEMMA 1. *With the above notations, if $q(x)$, $q(x)' \in \mathbf{Z}[x]$ then*
(i) $q(\gamma)f = 0 \Leftrightarrow q(\gamma) = 0$,
(ii) $q(\gamma) = 0 \Leftrightarrow p(x)$ *divides* $q(x)$ *over* $\mathbf{Z}$,
(iii) *if* $k > \mathrm{degree}(q(x))$ *and* $q(\gamma) = \gamma^k q(\gamma)'$, *then there exists* $q(x)'' \in \mathbf{Z}[x]$ *with* $\mathrm{degree}(q(x)'') \leqq r-1$ *such that* $q(\gamma) = \gamma^k q(\gamma)''$.

PROOF. (i) is obvious. (ii) If $q(\gamma) = 0$, then $p(x)$ divides $q(x)$ over $\mathbf{Q}$. Since $p(x)$ is primitive, it follows from Gauss' Lemma that $p(x)$ divides $q(x)$ over $\mathbf{Z}$. The converse is clear. (iii) Let s denote the degree of $q(x)'$. We shall give a proof for the case $s \geqq r$. By (ii), $p(x)$ divides $x^k q(x)' - q(x)$ over $\mathbf{Z}$. Hence there is a positive integer a_0 such that $a_1 = a_0 a_2$, where a_1 and a_2 are the leading coefficients of $x^k q(x)' - q(x)$ and $p(x)$, respectively. The degree of $q(x)'' = q(x)' - a_0 p(x) x^{s-r}$ is less than s and $q(\gamma) = \gamma^k q(\gamma)''$ holds. Repeating this process, we get the conclusion of (iii).

Let γ and f be as above. For a positive integer J, we define

$$V_J(j;\gamma) = \left\{ \sum_{m=0}^{j} k_m \gamma^m f : k_m \in \mathbf{Z}, |k_m| \leqq (m+J)^{20} \right\} \qquad (j \geqq 0).$$

PROPOSITION 3. *With the above notations, if all the eigenvalues of γ are not roots of unity, then for given $J > 0$ there exists an integer $K_1 > 0$ such that for all $K \geqq K_1$ and all $M > 0$, $V_J(K^2;\gamma) \cap \gamma^{K^2+K} V_J(M,\gamma) = \{0\}$.*

PROOF. We assume that $0 \neq \tilde{k} \in V_J(K^2;\gamma) \cap \gamma^{K^2+K} V_J(M;\gamma)$ for some K and some M, then there exist polynomials

$$a(x) = \sum_{m=0}^{K^2} k_m x^m \qquad (k_m \in \mathbf{Z} \text{ and } |k_m| \leqq (m+J)^{20}),$$

$$b(x) = \sum_{m=0}^{M} k_{m+K^2+K} x^m \qquad (k_{m+K^2+K} \in \mathbf{Z} \text{ and } |k_{m+K^2+K}| \leqq (m+J)^{20})$$

such that $\tilde{k} = a(\gamma)f = \gamma^{K^2+K} b(\gamma) f$. Hence it follows from Lemma 1 (i) that

$$(1) \qquad a(\gamma) = \gamma^{K2+K} b(\gamma).$$

Since $\gamma \in \mathrm{GL}(r, \mathbf{Q})$, there is a positive integer m_0 such that $m_0 q(\gamma) \in \mathrm{GL}(r, \mathbf{Z})$ for any $q(x) \in \mathbf{Z}[x]$ with degree $\leqq r-1$. Let γ act on $\mathbf{R}^r$, then $\mathbf{R}^r$ splits into a direct sum $\mathbf{R}^r = V_{-1} \oplus V_0 \oplus V_1$ of subspaces V_j such that V_j are γ-invariant and the eigenvalues of $\gamma_{|V_{-1}}$ have modulus <1, the eigenvalues of $\gamma_{|V_0}$ modulus one and the eigenvalues of $\gamma_{|V_1}$ modulus >1. We can find an integer $n_0 > 0$ such that $|\arg \lambda^{n_0}| < 1/2$ for all the eigenvalues λ of γ and $|\lambda^{n_0}| < 1/2$ for $|\lambda| < 1$ (by using Dirichlet's theorem to a rotation on an r-dimensional torus). Then we get easily that

$$|\lambda^{n_0} - 1| < \begin{cases} 1 - |\lambda^{n_0}|/2 & \text{if } |\lambda| < 1, \\ 1/2 & \text{if } |\lambda| = 1, \\ |\lambda^{n_0}| - 1/2 & \text{if } |\lambda| > 1. \end{cases}$$

Denote by I the identity matrix, then $\gamma^{n_0} - I$ is non-singular, since γ has no finite orbits except the zero vector of $\mathbf{R}^r$. For k with $0 < n_0 k < K$, operate $(\gamma^{n_0} - I)^k$ on both sides of the equality (1). Then by Lemma 1 (iii), there exists $p_0(x) \in \mathbf{Z}[x]$ with degree $(p_0(x)) \leqq r-1$ such that

$$(\gamma^{n_0} - I)^k a(\gamma) = \gamma^{K2+K} (\gamma^{n_0} - I)^k b(\gamma) = \gamma^{K2+K} p_0(\gamma).$$

Hence, by the choice of m_0 we get

$$(2) \qquad m_0 p_0(\gamma) = m_0 (\gamma^{n_0} - I)^k a(\gamma) \gamma^{-K2-K} = m_0 (\gamma^{n_0} - I)^k b(\gamma) \in \mathrm{GL}(r, \mathbf{Z}).$$

Therefore there exists a constant $C > 0$ depending only on the norm, the vector f and the integer m_0 such that $C < \|p_0(\gamma) f\|$. Since f splits uniquely the sum $f = f_{-1} + f_0 + f_1$ with some $f_i \in V_i$ $(i = -1, 0, 1)$, by (2) we have

$$(3) \qquad \begin{aligned} C &\leqq \|p_0(\gamma) f_{-1}\| + \|p_0(\gamma) f_0\| + \|p_0(\gamma) f_1\| \\ &\leqq \|(\gamma^{n_0} - I)^k b(\gamma) f_{-1}\| + \|(\gamma^{n_0} - I)^k \gamma^{-K2-K} a(\gamma) f_0\| + \|(\gamma^{n_0} - I)^k \gamma^{-K2-K} a(\gamma) f_1\| \end{aligned}$$

Let ρ_{-1} denote the minimum modulus of all the eigenvalues of $\gamma_{|V_{-1}}$, ξ be the maximum one of $\gamma_{|V_{-1}}$, θ be the minimum of $\gamma_{|V_1}$ and θ_1 be the maximum of $\gamma_{|V_1}$. Then it follows from the Jordan canonical form that there is a positive number d such that for all $m > 0$,

$$\|\gamma^m f_{-1}\| \leqq d m^r \xi^m \|f_{-1}\|,$$

$$\|\gamma^{-m} f_1\| \leqq d m^r \theta^{-m} \|f_1\|,$$

$$\|(\gamma^{n_0} - I)^m f_{-1}\| \leqq d m^r (1 - \rho_{-1}^{n_0}/2)^m \|f_{-1}\|,$$

$$\|(\gamma^{n_0} - I)^m f_0\| \leqq dm'2^{-m}\|f_0\|,$$

$$\|(I - \gamma^{-n_0})^m f_1\| \leqq dm'(1 - \theta_1^{-n_0}/2)^m \|f_1\|.$$

Take the integer part of K/n_0 as the integer k in (3). Then we can calculate easily that the last three terms of (3) tend to 0 as $K \to \infty$. The proof is completed.

§3. The splitting of compact abelian groups

Let X be a compact metric abelian group and σ be an automorphism of X. As before we denote by (G, γ) the dual of (X, σ). We say that (X, σ) satisfies *condition* (A) if for every $0 \neq g \in G$ there is a non-trivial polynomial $p(x) \in \mathbf{Z}[x]$ such that $p(\gamma)g = 0$, and that (X, σ) satisfies *condition* (B) if every $0 \neq g \in G$ has the condition that $p(\gamma)g \neq 0$ for all $0 \neq p(x) \in \mathbf{Z}[x]$. As before let K_g denote a subgroup $K_g = \Sigma_{-\infty}^{\infty} \gamma^j \langle g \rangle$ for $g \in G$.

The aim of this section is to prove the following

THEOREM 2. *Let X and σ be as above. Then X splits into a sum $X = X_1 + X_2 + X_3$ of exactly σ-invariant subgroups such that* (i) *X_1 is totally disconnected,* (ii) *X_2 is connected and satisfies condition* (A) *and* (iii) *X_3 is connected and satisfies condition* (B). *If in particular (X, σ) is ergodic, then X_i $(i = 1, 2, 3)$ is chosen such that (X_i, σ) is ergodic.*

The proof will be conducted using the following lemmas.

LEMMA 2. *Assume that G_1 is a γ-invariant torsion free subgroup of G. Then for any given $f \in G$ there exists an integer $d > 0$ such that $G_1 + dK_f$ is torsion free.*

PROOF. Assume that $G_1 + K_f$ is not torsion free, then there is a primitive polynomial $p(x)$ with minimum degree such that for some $d > 0$, $dp(\gamma)f \in G_1$. We show that this d is the desired one. Assume that $mg = 0$ for some $g \in G_1 + dK_f$ and some integer $m > 0$. Then there are $g_1 \in G_1$, $b > 0$ and $q(x) \in \mathbf{Z}[x]$ such that $g = g_1 + d\gamma^{-b}q(\gamma)f$. Hence $mdq(\gamma)f = -m\gamma^b g_1 \in G_1$. By Gauss' lemma it follows that $q(x) = q(x)'p(x)$ for some $q(x)' \in \mathbf{Z}[x]$, and hence $g = g_1 + d\gamma^{-b}q(\gamma)'p(\gamma)f$ belongs to G_1. Since G_1 is torsion free, we get $g = 0$, which implies that $G_1 + dK_f$ is torsion free.

LEMMA 3. *Let X_0 be the connected component of 0 in X. Then X contains an exactly σ-invariant totally disconnected subgroup X_1 such that $X = X_0 + X_1$.*

PROOF. Denote by G' the maximum torsion subgroup of G. For a character $g_0 \notin G'$, it follows that there is an integer $d_0 > 0$ such that $d_0 K_{g_0}$ is torsion free.

Since X is metrizable, G must be countable. Using Lemma 2 inductively, we see that there exist positive integers $d_1, d_2, \cdots$ and characters $g_1, g_2, \cdots \not\in G'$ such that $G'' = \Sigma_{j=0}^{\infty} d_j K_{g_j}$ is torsion free and G/G'' is a torsion group. Let X_1 denote the annihilator of G'' in X. Then X_1 has the dual group G/G''. Thus X_1 is totally disconnected and exactly σ-invariant. Since X/X_0 is totally disconnected, $X/(X_0 + X_1)$ must be connected and totally disconnected, i.e. $X = X_0 + X_1$.

LEMMA 4. *Let Y be a compact connected metric abelian group and $\tilde{\sigma}$ be an automorphism of Y. Then Y splits into a sum $Y = Y_2 + Y_3$ of exactly $\tilde{\sigma}$-invariant connected subgroups Y_2 and Y_3 satisfying* (ii) *and* (iii) *of Theorem* 2, *respectively.*

PROOF. As before let $(\tilde{G}, \tilde{\sigma})$ be the dual of $(Y, \tilde{\sigma})$. We denote by $\tilde{G}_A$ the maximum subgroup of $\tilde{G}$ satisfying condition (A). If $g \not\in \tilde{G}_A$, then $K_g = \Sigma_{-\infty}^{\infty} \tilde{\gamma}^j \langle g \rangle$ has a direct sum splitting $K_g = \bigoplus_{-\infty}^{\infty} \tilde{\gamma}^j \langle g \rangle$ (the notation $\bigoplus_{-\infty}^{\infty} G_n$ used here means the restricted direct sum for an infinite family of subgroups G_n). For $g_{i_1} \not\in \tilde{G}_A$, we denote by g_{i_2} a character $f \not\in \tilde{G}_A$ such that $K_{g_{i_1}} \cap K_f = \{0\}$, and by g_{i_3} a character $h \not\in \tilde{G}_A$ such that $(K_{g_{i_1}} \oplus K_{g_{i_2}}) \cap K_h = \{0\}$. Repeating this step, we get a sequence $\{K_{g_{i_n}}\}$ of subgroups such that $\tilde{G}_B = \bigoplus_{n \geqq 1} K_{g_{i_n}}$ is a subgroup of $\tilde{G}$, $\tilde{G}_A \cap \tilde{G}_B = \{0\}$ and every $0 \neq \dot{g} \in \tilde{G}_A / \tilde{G}_B$ satisfies the condition that $p(\tilde{\gamma})\dot{g} = 0$ for some $0 \neq p(x) \in \mathbf{Z}[x]$. Let us put $\tilde{G}'_B = \{f \in \tilde{G} : mf \in \tilde{G}_B$ for some $m \neq 0\}$, then $\tilde{G}/\tilde{G}'_B$ is torsion free and also $\tilde{G}/\tilde{G}_A$ is so. Hence the annihilator Y_2 of $\tilde{G}'_B$ in Y and the annihilator Y_3 of $\tilde{G}_A$ in Y are connected. It is easy to check that $Y = Y_2 + Y_3$, and Y_2 and Y_3 satisfy (ii) and (iii) of Theorem 2, respectively.

From Lemmas 3 and 4 we get the conclusion of the first statement of Theorem 2. The second statement will be obtained by Lemma 3 and the following lemma.

LEMMA 5. *Let σ be an ergodic automorphism of X. Assume that W_i $(i = 1, 2)$ are exactly σ-invariant subgroups such that $X = W_1 + W_2$. Then there exists an exactly σ-invariant subgroup W_3 of W_2 such that (W_3, σ) is ergodic and X is expressed as $X = W_1 + W_3$.*

PROOF. It is known (cf. [19] or p. 242 of [10]) that there is a σ-invariant subgroup W_3 such that (W_3, σ) is ergodic and $(W_2/W_3, \sigma)$ has zero entropy. Since $W_2/(W_1 \cap W_2)$ is algebraically isomorphic to X/W_1, $(W_2/(W_1 \cap W_2), \sigma)$ is a factor of the ergodic system (X, σ). Hence $(W_2/(W_1 \cap W_2), \sigma)$ is ergodic and by [19] a K-system. Since $(W_2/(W_3 + (W_1 \cap W_2)), \sigma)$ is a factor of the system $(W_2/(W_1 \cap W_2), \sigma)$, its entropy is zero and positive if it is not trivial. Hence $X = W_1 + W_3$.

Using Theorem 2 and Ornstein's theorem [17], we get the following

COROLLARY. *If σ is an ergodic automorphism of a compact metric abelian group X, then (X, σ) is Bernoullian.*

This is a combination of the following known Lemmas 7, 8 and 9. A shift automorphism is called a *simple Bernoulli automorphism* when the state space is an algebraic simple group with Haar measure.

LEMMA 6. *Let X be a compact totally disconnected metric abelian group. If σ is an ergodic automorphism of X, then X contains a sequence $X = F_0 \supset F_1 \supset \cdots$ of σ-invariant subgroups such that $\bigcap F_n = \{0\}$ and for every $n \geqq 0$, there is a decreasing sequence $\{F_{n,i}\}$ of σ-invariant subgroups such that $\bigcap_i F_{n,i} = F_{n+1}$ and for every $i \geqq 1$, $\sigma_{|F_n/F_{n,i}}$ is a simple Bernoulli automorphism.*

The lemma is shown in [1], so we omit the proof.

LEMMA 7. *Let X be as in Lemma 6. If σ is an ergodic automorphism of X, then (X, σ) is Bernoullian.*

PROOF. Let $\{F_n\}$ be a sequence of subgroups satisfying all the conditions of Lemma 6. Let S_i be a skew product transformation of $X/F_n \times F_n/F_{n,i}$ induced by $\sigma_{|X/F_n}$ and $\sigma_{|F_n/F_{n,i}}$ for $i \geqq 1$. Then S_i is metrically isomorphic to $\sigma_{|X/F_{n,i}}$ and $\sigma_{|F_n/F_{n,i}}$ is a simple Bernoulli automorphism. We have that $(X/F_n, \sigma)$ is Bernoullian (cf. p. 208 of [11]), and hence so is (X, σ).

LEMMA 8. *Let X be a compact connected metric abelian group and σ be an ergodic automorphism of X. Assume that (X, σ) satisfies condition* (A), *then (X, σ) is Bernoullian.*

PROOF. Let $\{G_n\}$ be a sequence $G_1 \subset G_2 \subset \cdots \subset \bigcup_n G_n = G$ of γ-invariant subgroups such that for every $n \geqq 1$ the rank of G_n is finite. If X_n denotes the annihilator of G_n in X for $n \geqq 1$, then we have that X/X_n is a solenoidal group, so that $(X/X_n, \sigma)$ is Bernoullian (by Theorem 1), and hence so is (X, σ).

LEMMA 9. *Let X and σ be as in Lemma 8. Assume that (X, σ) satisfies condition* (B). *Then (X, σ) is Bernoullian.*

PROOF. As before let (G, γ) be the dual of (X, σ). Since G is countable, there is a sequence $G_1 \subset G_2 \subset \cdots \subset \bigcup_n G_n = G$ of exactly γ-invariant subgroups G_n such that $G_n = \Sigma_{i=1}^n K_{f_i}$ $(f_i \in G)$ for $n \geqq 1$. Let X_n be the annihilator of G_n in X for $n \geqq 1$, then $X_n \searrow \{0\}$ and X/X_n has the dual group G_n. It is known (p. 167 of [8]) that there is the minimum divisible extension $(\bar{G}_n, \gamma)$ of (G_n, γ). Since $\bar{G}_n$ is divisible and torsion free, we can consider $\bar{G}_n$ to be a $\mathbf{Q}[x, x^{-1}]$-module. Since

$\mathbf{Q}[x, x^{-1}]$ is a principal ideal domain, there are elements $g_1, \cdots, g_p \in G_n$ such that $\bar{G}_n = \bigoplus_{i=1}^{p} \mathbf{Q}[\gamma, \gamma^{-1}]g_i$ (cf. p. 85, theorem 2 in ch. 7 of [4]). Hence the dual of $(\bar{G}_n, \gamma)$ is clearly Bernoullian, so that $(X/X_n, \sigma)$ is also Bernoullian. Since n is arbitrary, we get the conclusion.

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References

1. N. Aoki, *Zero-dimensional automorphisms having a dense orbit*, J. Math. Soc. Japan, to appear.

2. N. Aoki and I. Kubo, *A number theoretical lemma and its application to the ergodic theory*, preprint.

3. N. Aoki and H. Totoki, *Ergodic automorphisms of T^∞ are Bernoulli transformations*, Publ. Res. Inst. Math. Sci. **10** (1975), 535–544.

4. N. Bourbaki, Elements de Algèbre Mathématique, Hermann, Paris, 1964.

5. N. A. Friedman and D. S. Ornstein, *On isomorphism of weak Bernoulli transformations*, Advances in Math. **5** (1971), 365–394.

6. Y. Katznelson, *Ergodic automorphisms of T^n are Bernoulli shifts*, Israel J. Math. **10** (1971), 186–195.

7. I. Kubo, H. Murata and H. Totoki, *On the isomorphism problem for endomorphisms of Lebesgue spaces I, II and III*, Publ. Res. Inst. Math. Sci. **2** (1974), 285–317.

8. A. G. Kursch, *The Theory of Groups I, II*, Chelsea, New York, 1960.

9. S. Lang, *Algebra*, Addison-Wesley, 1972.

10. D. Lind, *The structure of skew products with ergodic group automorphisms*, Israel J. Math. **28** (1977), 205–248.

11. G. Miles and K. Thomas, *The breakdown of automorphisms of compact topological groups*, in *Studies in Probability and Ergodic Theory*, Advances in Math. Supplementary Studies, Vol. 2, 1978, pp. 207–218.

12. G. Miles and K. Thomas, *On the polynomial uniformity of translations of the n-torus*, in *Studies in Probability and Ergodic Theory*, Advances in Math. Supplementary Studies, Vol. 2, 1978, pp. 219–229.

13. G. Miles and K. Thomas, *Generalized torus automorphisms are Bernoullian*, in *Studies in Probability and Ergodic Theory*, Advances in Math. Supplementary Studies, Vol. 2, 1978, pp. 231–249.

14. D. S. Ornstein, *Imbedding Bernoulli shifts in flows*, in *Contribution to Ergodic Theory and Probability*, Springer-Verlag, 1970, pp. 178–218.

15. D. S. Ornstein, *Bernoulli shifts with the same entropy are isomorphic*, Advances in Math. **4** (1970), 337–452.

16. D. S. Ornstein, *Two Bernoulli shifts with infinite entropy are isomorphic*, Advances in Math. **5** (1971), 339–348.

17. D. S. Ornstein, *Factors of Bernoulli shifts are Bernoulli shifts*, Advances in Math. **5** (1971), 349–364.

18. L. Pontrjagin, *Topological Groups*, Princeton Univ. Press, Princeton, 1946.

19. V. A. Rohlin, *Metric properties of endomorphisms of compact commutative groups*, Amer. Math. Soc. Transl. **64** (1967), 244–272.

20. B. Weiss, *The isomorphisms problem in the ergodic theory*, Bull. Amer. Math. Soc. **78** (1972), 668–684.

21. S. A. Yuzvinskii, *Metric properties of endomorphisms of compact groups*, Amer. Math. Soc. Transl. **66** (1968), 63–98.

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